Hilbert's 14^{th} Problem

Question: Let k be a field, $k[X_1, X_2, \cdots, X_n]$ a polynomial ring in n variables over k and L a subfield of $k(X_1, X_2, \cdots, X_n)$ containing k. Is $L \cap k[X_1, X_2, \cdots, X_n]$ a finitely generated k-algebra?

Hilbert's 14^{th} Problem

Question: Let k be a field, $k[X_1, X_2, \cdots, X_n]$ a polynomial ring in n variables over k and L a subfield of $k(X_1, X_2, \cdots, X_n)$ containing k. Is $L \cap k[X_1, X_2, \cdots, X_n]$ a finitely generated k-algebra?
YES, if tr. $\text{deg.}_k L \leq 2$ (Zariski, 1954)

Hilbert's 14^{th} Problem

Question: Let k be a field, $k[X_1, X_2, \cdots, X_n]$ a polynomial ring in n variables over k and L a subfield of $k(X_1, X_2, \cdots, X_n)$ containing k. Is $L \cap k[X_1, X_2, \cdots, X_n]$ a finitely generated k-algebra?

YES, if tr. $\text{deg.}_k L \leq 2$ (Zariski, 1954)

NO, if tr. $\text{deg.}_k L \geq 3$ (Kuroda, 2005)

Hilbert's 14^{th} Problem

Question: Let k be a field, $k[X_1, X_2, \cdots, X_n]$ a polynomial ring in n variables over k and L a subfield of $k(X_1, X_2, \cdots, X_n)$ containing k. Is $L \cap k[X_1, X_2, \cdots, X_n]$ a finitely generated k-algebra?

YES, if tr. $\deg._k L \leq 2$ (Zariski, 1954)

NO, if tr. $\deg._k L \geq 3$ (Kuroda, 2005)

Question: Let G be a linear algebraic group acting on an affine algebraic variety X. Is the ring of invariants $k[X]^G = \{f \in k[X] | f(g.x) = f(x)\ \forall g \in G, x \in X\}$ a finitely generated k algebra?

Hilbert's 14^{th} Problem

Question: Let k be a field, $k[X_1, X_2, \cdots, X_n]$ a polynomial ring in n variables over k and L a subfield of $k(X_1, X_2, \cdots, X_n)$ containing k. Is $L \cap k[X_1, X_2, \cdots, X_n]$ a finitely generated k-algebra?
YES, if tr. $\deg._k L \leq 2$ (Zariski, 1954)
NO, if tr. $\deg._k L \geq 3$ (Kuroda, 2005)
Question: Let G be a linear algebraic group acting on an affine algebraic variety X. Is the ring of invariants
$k[X]^G = \{f \in k[X] | f(g.x) = f(x) \ \forall g \in G, x \in X\}$ a finitely generated k algebra?
Yes, if G is reductive (Hilbert's Finiteness Theorem).

Hilbert's 14^{th} Problem

Question: Let k be a field, $k[X_1, X_2, \cdots, X_n]$ a polynomial ring in n variables over k and L a subfield of $k(X_1, X_2, \cdots, X_n)$ containing k. Is $L \cap k[X_1, X_2, \cdots, X_n]$ a finitely generated k-algebra?
YES, if tr. deg.$_k L \leq 2$ (Zariski, 1954)
NO, if tr. deg.$_k L \geq 3$ (Kuroda, 2005)
Question: Let G be a linear algebraic group acting on an affine algebraic variety X. Is the ring of invariants $k[X]^G = \{f \in k[X] | f(g.x) = f(x) \ \forall g \in G, x \in X\}$ a finitely generated k algebra?
Yes, if G is reductive (Hilbert's Finiteness Theorem).
NO, $G = G_a^{13}$, $X = \mathbb{A}^{32}$ (Nagata, 1960)

$\mathbb{G}_a$ action on Affine space $\mathbb{A}_k^n$

Let k be a field of characteristic 0.
$\mathbb{G}_a$ action on Affine space $\mathbb{A}_k^n \longleftrightarrow D \in LND(k^{[n]})$
Ring of invariants$=$ ker $(D) = \{f \in k^{[n]} | D(f) = 0\}$.
Question: Given a $D \in \text{LND}(k^{[n]})$, is $\ker(D)$ a finitely generated k-algebra?

$\mathbb{G}_a$ action on Affine space $\mathbb{A}_k^n$

Let k be a field of characteristic 0.
$\mathbb{G}_a$ action on Affine space $\mathbb{A}_k^n \longleftrightarrow D \in LND(k^{[n]})$
Ring of invariants= ker $(D) = \{f \in k^{[n]} | D(f) = 0\}$.
Question: Given a $D \in \text{LND}(k^{[n]})$, is ker$(D)$ a finitely generated k-algebra?
Yes , if

1. D is linear (Maurer-Weitzenböck's Theorem), (Weitzenböck, 1932; Sheshadri, 1962)

$\mathbb{G}_a$ action on Affine space $\mathbb{A}_k^n$

Let k be a field of characteristic 0.
$\mathbb{G}_a$ action on Affine space $\mathbb{A}_k^n \longleftrightarrow D \in LND(k^{[n]})$
Ring of invariants= $\ker(D) = \{f \in k^{[n]} | D(f) = 0\}$.
Question: Given a $D \in \text{LND}(k^{[n]})$, is $\ker(D)$ a finitely generated k-algebra?
Yes , if

1. D is linear (Maurer-Weitzenböck's Theorem), (Weitzenböck, 1932; Sheshadri, 1962)
2. $n = 2, 3$ (Rentschler 1968, Miyanishi, 1985)

$\mathbb{G}_a$ action on Affine space $\mathbb{A}_k^n$

Let k be a field of characteristic 0.
$\mathbb{G}_a$ action on Affine space $\mathbb{A}_k^n \longleftrightarrow D \in LND(k^{[n]})$
Ring of invariants$=$ ker $(D) = \{f \in k^{[n]} | D(f) = 0\}$.
Question: Given a $D \in \mathrm{LND}(k^{[n]})$, is $\ker(D)$ a finitely generated k-algebra?
Yes , if

1. D is linear (Maurer-Weitzenböck's Theorem), (Weitzenböck, 1932; Sheshadri, 1962)
2. $n = 2, 3$ (Rentschler 1968, Miyanishi, 1985)
3. $n = 4$ and D is monomial LND (Kuroda, 2003)

$\mathbb{G}_a$ action on Affine space $\mathbb{A}^n_k$

Let k be a field of characteristic 0.
$\mathbb{G}_a$ action on Affine space $\mathbb{A}^n_k \longleftrightarrow D \in LND(k^{[n]})$
Ring of invariants= ker $(D) = \{f \in k^{[n]} | D(f) = 0\}$.
Question: Given a $D \in \text{LND}(k^{[n]})$, is $\ker(D)$ a finitely generated k-algebra?
Yes , if

1. D is linear (Maurer-Weitzenböck's Theorem), (Weitzenböck, 1932; Sheshadri, 1962)
2. $n = 2, 3$ (Rentschler 1968, Miyanishi, 1985)
3. $n = 4$ and D is monomial LND (Kuroda, 2003)
4. $n = 4$ and D is triangular (Daigle and Freudenburg, 2001)
5. $n = 4$ and $\text{rank}(D) \leq 3$(Bhatwadekar and Daigle, 2009)

$\mathbb{G}_a$ action on Affine space $\mathbb{A}_k^n$

Let k be a field of characteristic 0.
$\mathbb{G}_a$ action on Affine space $\mathbb{A}_k^n \longleftrightarrow D \in LND(k^{[n]})$
Ring of invariants= ker $(D) = \{f \in k^{[n]} | D(f) = 0\}$.
Question: Given a $D \in \text{LND}(k^{[n]})$, is $\ker(D)$ a finitely generated k-algebra?
Yes , if

1. D is linear (Maurer-Weitzenböck's Theorem), (Weitzenböck, 1932; Sheshadri, 1962)
2. $n = 2, 3$ (Rentschler 1968, Miyanishi, 1985)
3. $n = 4$ and D is monomial LND (Kuroda, 2003)
4. $n = 4$ and D is triangular (Daigle and Freudenburg, 2001)
5. $n = 4$ and $\text{rank}(D) \leq 3$(Bhatwadekar and Daigle, 2009)
6. $D(s) = 1$ for some $s \in k^{[n]}$ (Slice theorem)

No,

1. For $n = 7$ (Robert, 1990)

   $$D = X^{t+1}\frac{\partial}{\partial S} + Y^{t+1}\frac{\partial}{\partial T} + Z^{t+1}\frac{\partial}{\partial U} + (XYZ)^t\frac{\partial}{\partial V}, \text{ for } t \geq 2.$$

continued...

No,

1. For $n = 7$ (Robert, 1990)

$$D = X^{t+1}\frac{\partial}{\partial S} + Y^{t+1}\frac{\partial}{\partial T} + Z^{t+1}\frac{\partial}{\partial U} + (XYZ)^t\frac{\partial}{\partial V}, \text{ for } t \geq 2.$$

2. n=6 (Freudenburg, 2000)

$$D = X^3\frac{\partial}{\partial S} + Y^3 S\frac{\partial}{\partial T} + Y^3 T\frac{\partial}{\partial U} + (XY)^2\frac{\partial}{\partial V}$$

continued...

No,

1. For $n = 7$ (Robert, 1990)
   $$D = X^{t+1}\frac{\partial}{\partial S} + Y^{t+1}\frac{\partial}{\partial T} + Z^{t+1}\frac{\partial}{\partial U} + (XYZ)^t\frac{\partial}{\partial V}, \text{ for } t \geq 2.$$
2. n=6 (Freudenburg, 2000)
   $$D = X^3\frac{\partial}{\partial S} + Y^3 S\frac{\partial}{\partial T} + Y^3 T\frac{\partial}{\partial U} + (XY)^2\frac{\partial}{\partial V}$$

continued...

No,

1. For $n = 7$ (Robert, 1990)
$$D = X^{t+1}\frac{\partial}{\partial S} + Y^{t+1}\frac{\partial}{\partial T} + Z^{t+1}\frac{\partial}{\partial U} + (XYZ)^t\frac{\partial}{\partial V}, \text{ for } t \geq 2.$$
2. n=6 (Freudenburg, 2000)
$$D = X^3\frac{\partial}{\partial S} + Y^3 S\frac{\partial}{\partial T} + Y^3 T\frac{\partial}{\partial U} + (XY)^2\frac{\partial}{\partial V}$$
3. n=5, (Daigle and Freudenburg, 1999)
$$D = X^3\frac{\partial}{\partial S} + S\frac{\partial}{\partial T} + T\frac{\partial}{\partial U} + X^2\frac{\partial}{\partial V}$$

No,

1. For $n = 7$ (Robert, 1990)

$$D = X^{t+1}\frac{\partial}{\partial S} + Y^{t+1}\frac{\partial}{\partial T} + Z^{t+1}\frac{\partial}{\partial U} + (XYZ)^t\frac{\partial}{\partial V}, \text{ for } t \geq 2.$$

2. n=6 (Freudenburg, 2000)

$$D = X^3\frac{\partial}{\partial S} + Y^3 S\frac{\partial}{\partial T} + Y^3 T\frac{\partial}{\partial U} + (XY)^2\frac{\partial}{\partial V}$$

3. n=5, (Daigle and Freudenburg, 1999)

$$D = X^3\frac{\partial}{\partial S} + S\frac{\partial}{\partial T} + T\frac{\partial}{\partial U} + X^2\frac{\partial}{\partial V}$$

Note: D is $k[X, S - XV]$-derivation of $k[X, S - XV][V, T, U]$. It can be re-written as

$$D = X^2\frac{\partial}{\partial V} + ((S - XV) + XV)\frac{\partial}{\partial T} + T\frac{\partial}{\partial U}$$

No,

1. For $n = 7$ (Robert, 1990)
$$D = X^{t+1}\frac{\partial}{\partial S} + Y^{t+1}\frac{\partial}{\partial T} + Z^{t+1}\frac{\partial}{\partial U} + (XYZ)^t\frac{\partial}{\partial V}, \text{ for } t \geq 2.$$
2. n=6 (Freudenburg, 2000)
$$D = X^3\frac{\partial}{\partial S} + Y^3 S\frac{\partial}{\partial T} + Y^3 T\frac{\partial}{\partial U} + (XY)^2\frac{\partial}{\partial V}$$
3. n=5, (Daigle and Freudenburg, 1999)
$$D = X^3\frac{\partial}{\partial S} + S\frac{\partial}{\partial T} + T\frac{\partial}{\partial U} + X^2\frac{\partial}{\partial V}$$
Note: D is $k[X, S - XV]$-derivation of $k[X, S - XV][V, T, U]$.
It can be re-written as
$$D = X^2\frac{\partial}{\partial V} + ((S - XV) + XV)\frac{\partial}{\partial T} + T\frac{\partial}{\partial U}$$

Daigle-Freudenburg example is a triangular monomial derivation of $k^{[5]}$ and also a triangular derivation of $R^{[3]}$, for $R = k^{[2]}$.

How to find Kernel of a LND

Let $B = k[X_1, X_2, \cdots, X_n], D \in \text{LND}(B), A = \ker(D)$ and $r \in B$ be a local slice of D (i.e., $f = D(r) \neq 0, D^2(r) = 0$).
Dixmier Map: $\pi_r(g) = \sum_{i \geq 0} \frac{(-1)^i}{i!} \frac{r^i}{f^i} D^i(g)$.
Then $\pi : B \to A_f$ is a surjective ring homomorphism.

How to find Kernel of a LND

Let $B = k[X_1, X_2, \cdots, X_n], D \in \text{LND}(B), A = \ker(D)$ and $r \in B$ be a local slice of D (i.e., $f = D(r) \neq 0, D^2(r) = 0$).

Dixmier Map: $\pi_r(g) = \sum_{i \geq 0} \frac{(-1)^i}{i!} \frac{r^i}{f^i} D^i(g)$.

Then $\pi : B \to A_f$ is a surjective ring homomorphism.

1. Van-den-Essen's Algorithm

Let $A_0 = k[f^{m_1}\pi(X_1), f^{m_2}\pi(X_2), \cdots f^{m_n}\pi(X_n), f]$.

Construct A_i Inductively as follows

$A_i = \{h \in B | fh \in A_{n-1}\}$

$A = A_n$ iff $A_n = A_{n+1}$

How to find Kernel of a LND

Let $B = k[X_1, X_2, \cdots, X_n], D \in \text{LND}(B), A = \ker(D)$ and $r \in B$ be a local slice of D (i.e., $f = D(r) \neq 0, D^2(r) = 0$).

Dixmier Map: $\pi_r(g) = \sum_{i \geq 0} \frac{(-1)^i}{i!} \frac{r^i}{f^i} D^i(g)$.

Then $\pi : B \to A_f$ is a surjective ring homomorphism.

1. Van-den-Essen's Algorithm

Let $A_0 = k[f^{m_1}\pi(X_1), f^{m_2}\pi(X_2), \cdots f^{m_n}\pi(X_n), f]$.

Construct A_i Inductively as follows

$A_i = \{h \in B | fh \in A_{n-1}\}$

$A = A_n$ iff $A_n = A_{n+1}$

Some Positive Results

Theorem (-, Singha)

Let R be a UFD and D a triangular monomial derivation of $B = R[X, Y, Z]$ defined by

$$D(X) = f, \ D(Y) = gX^a \text{ and } D(Z) = hX^bY^c$$

where $f, g, h \in R$, f, g, h are pairwise co-prime and a, b, c are non-negative integers. Then $\ker(D)$ is generated by at most 3 elements over R.

Some Positive Results

Theorem (-, Singha)

Let R be a UFD and D a triangular monomial derivation of $B = R[X, Y, Z]$ defined by

$$D(X) = f,\ D(Y) = gX^a \text{ and } D(Z) = hX^bY^c$$

where $f, g, h \in R$, f, g, h are pairwise co-prime and a, b, c are non-negative integers. Then $\ker(D)$ is generated by at most 3 elements over R.

$\ker(D) = R[F, G, H]$, where $F = -gX^{(a+1)} + (a+1)fY$,
$G = h\sum_{i=0}^{c}(-1)^{i+1}\alpha_i f^i g^{(c-i)} X^{b+(a+1)c+1-i(a+1)} Y^i + (-1)^{c+2}\alpha_c(b+1)f^{(c+1)}Z$,
$H = \frac{1}{f^\gamma}(h^{t_3}F^{t_1} + (-1)^{t_1+t_3+1}g^{t_1-ct_3}G^{t_3})$,
where $\alpha_0 = 1$, $\alpha_i = \frac{\prod_{j=0}^{i}(b+(a+1)c+1-(j-1)(a+1))}{i!}$ for $i = 1, \ldots, c$,
$t_1 = \frac{b+(a+1)c+1}{d}$, $t_3 = \frac{a+1}{d}$ and $d = \gcd(a+1, b+(a+1)c+1)$.

Theorem (-, Singha)

Let R be a UFD containing $\mathbb{Q}$, $B = R[X, Y, Z]$ and D_0 an irreducible triangular derivation of $B_0 = R[X, Y]$ such that

$$D_0(X) = \alpha \in R, \; D_0(Y) = f(X) \in R[X].$$

Then $\ker(D_0) = R[F]$. Extend the derivation D_0 to $B = B_0[Z]$ by defining $D(Z) = F^m$. Then $\ker(D)$ is generated by at most 3 elements over R.

Theorem (-, Singha)

Let R be a UFD containing $\mathbb{Q}$, $B = R[X, Y, Z]$ and D_0 an irreducible triangular derivation of $B_0 = R[X, Y]$ such that

$$D_0(X) = \alpha \in R, \; D_0(Y) = f(X) \in R[X].$$

Then $\ker(D_0) = R[F]$. Extend the derivation D_0 to $B = B_0[Z]$ by defining $D(Z) = F^m$. Then $\ker(D)$ is generated by at most 3 elements over R.

Let $f(X) = \sum_{i=0}^{n} a_i X^i$.

Then $F = \alpha Y + \sum_{i=1}^{n+1} b_i X^i$, where $b_1 = -a_0$ and

$$b_k = \{-1 + \sum_{i=1}^{k-1} \frac{(-1)^{i+1}}{(i+1)!} \prod_{j=1}^{i} (n-j+1)\} a_{k-1}, \text{ for } k = 2, \cdots, n+1.$$

$G = -\alpha Z + XF^m$,
$H = \frac{1}{\alpha}(F^{m(m(n+1)+1)} - \sum_{i=m}^{m(n+1)} c_i F^{m(m(n+1)-i)} G^i)$,
where c_i is the coefficient of X^i in g^m for $i = m, \cdots, m(n+1)$.
$\ker(D) = R[F, G, H]$.

Theorem

Let R be a UFD containing $\mathbb{Q}$ and D_0 a derivation of $B_0 = R[X, Y]$ defined by

$$D_0(X) = f, D_0(Y) = g$$

where $f, g \in R$. Choose $h \in \ker(D_0)$. Extend the derivation D_0 to $B = B_0[Z]$ by defining $D(Z) = h$. Then $\ker(D)$ is generated by at most 3 elements over R.

$\ker(D) = R[gX - fY, -hX + fZ, hY - gZ]$

Theorem

Let R be a UFD containing $\mathbb{Q}$ and D a nice derivation of $B = R[X, Y, Z]$ such that

$$D(X) = f, D(Y) = g, D(Z) = h,$$

where $f \in R$ and f, g, h are pairwise co-prime. Then $\ker(D)$ is generated by at most 3 elements over R.

$\ker(D) = R[gX - fY, -hX + fZ, hY - gZ]$.

Theorem (Joseph Khoury, 2001)

If $R = k[U, V] = k^{[2]}$ *and* $a_1, \cdots, a_m$ *for* $m \geq 1$ *are monomials in* U, V *(not all zero), then the kernel of the elementary derivation* $D = \sum_{i=1}^{m} a_i \frac{\partial}{\partial Y_i}$ *of* $B = R[Y_1, \cdots, Y_m]$ *is a polynomial ring in* $m-1$ *variables over* R*. Moreover, the* $m-1$ *generators can be chosen to be of the form* $b_i Y_i - b_j Y_j$*, where* $b_i, b_j \in R$ *for* $1 \leq i < j \leq m$*.*

Theorem (-, Singha)

Let $R = k[U, V]$ and D be the derivation of $R[X, Y, Z]$ defined by $D(X) = f(U, V)$, $D(Y) = g(U, V)$ and $D(Z) = \sum_{i+j=n} H_{(i,j)} X^i Y^j$, where $H_{(i,j)} \in k[U, V]$ and $gcd(f, g) = 1$. Then $\ker(D)$ is generated by at most 3 elements over R.

$\ker(D) = R[f_3, f_4, f_5]$, where

$f_3 = -gX + fY$;

$$\mathrm{f}_4 = f^{n+1}(Z - H_{(n,0)} \frac{1}{n+1} \frac{1}{f} - H_{(n-1,1)} \{ \frac{1}{n} \frac{1}{f} X^n Y - \frac{1}{n(n+1)} \frac{1}{f^2} g X^{n+1} \}$$

$$+ \sum_{j \geq 2; i+j=n} \sum_{k=1}^{j} (-1)^{k+1} \frac{\prod_{l=1}^{k} (j-l+1)}{\prod_{l=0}^{k} (i+l+1)} \frac{1}{f^{k+1}} g^k X^{i+k+1} Y^{j-k});$$

$$\mathrm{f}_5 = \frac{1}{f}(g f_4 - (-1)^{n+1} \frac{H_{(0,n)}}{n+1} f_3^{n+1}).$$

Generalization of Daigle-Freudenburg Counterexample

$$D = X^3\frac{\partial}{\partial S} + S\frac{\partial}{\partial T} + T\frac{\partial}{\partial U} + X^2\frac{\partial}{\partial V}$$

Generalization of Daigle-Freudenburg Counterexample

$$D = X^3 \frac{\partial}{\partial S} + S \frac{\partial}{\partial T} + T \frac{\partial}{\partial U} + X^2 \frac{\partial}{\partial V}$$

Theorem

Let k be a field of characteristic zero and $B = k[Y_1, \ldots, Y_l, X, S, T, U, V]$, where $l \geq 0$. Let

$$D = f_1 g(X)^m \partial_S + f_2 S \partial_T + f_3 T \partial_U + f_4 g(X)^r \partial_V,$$

where $g(X) \in k[X] \setminus k$, $m, r \in \mathbb{N}$, $\frac{2m}{3} \leq r \leq m-1$, and $f_i (\neq 0) \in k[Y_1, \ldots, Y_l]$. For $m \geq 3$, $\ker D$ is not a finitely generated k-algebra.

Generalization of Daigle-Freudenburg Counterexample

$$D = X^3\frac{\partial}{\partial S} + S\frac{\partial}{\partial T} + T\frac{\partial}{\partial U} + X^2\frac{\partial}{\partial V}$$

Theorem

Let k be a field of characteristic zero and $B = k[Y_1, \dots, Y_l, X, S, T, U, V]$, where $l \geq 0$. Let

$$D = f_1 g(X)^m \partial_S + f_2 S \partial_T + f_3 T \partial_U + f_4 g(X)^r \partial_V,$$

where $g(X) \in k[X] \setminus k$, $m, r \in \mathbb{N}$, $\frac{2m}{3} \leq r \leq m-1$, and $f_i(\neq 0) \in k[Y_1, \dots, Y_l]$. For $m \geq 3$, $\ker D$ is not a finitely generated k-algebra.

Theorem

Let $B := k[U, Y_1, Y_2, \cdots, Y_{m+1}, V]$ for $m \geq 3$, $D_0 := U^t\frac{\partial}{\partial Y_1} + Y_1\frac{\partial}{\partial Y_2} + \cdots + Y_m\frac{\partial}{\partial Y_{m+1}}$ for $t \geq 2$ and $D := D_0 + U^r\partial_V$. If $\frac{t}{2} \leq r \leq t-1$, then $\ker(D)$ is not a finitely generated k-algebra.